Degree distribution and assortativity in line graphs of complex networks

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Abstract

Topological characteristics of links of complex networks influence the dynamical processes executed on networks triggered by links, such as cascading failures triggered by links in power grids and epidemic spread due to link infection. The line graph transforms links in the original graph into nodes. In this paper, we investigate how graph metrics in the original graph are mapped into those for its line graph. In particular, we study the degree distribution and the assortativity of a graph and its line graph. Specifically, we show, both analytically and numerically, the degree distribution of the line graph of an Erdős-Rényi graph follows the same distribution as its original graph. We derive a formula for the assortativity of line graphs and indicate that the assortativity of a line graph is not linearly related to its original graph. Additionally, line graphs of various graphs, e.g. Erdős-Rényi graphs, scale-free graphs, show positive assortativity. In contrast, we find certain types of trees and non-trees whose line graphs have negative assortativity.

Keywords: Degree distribution, assortativity, line graph, complex network

1. Introduction

Infrastructures, such as the Internet, electric power grids and transportation networks, are crucial to modern societies. Most researches focus on the robustness of such networks to node failures [1, 2]. Specifically, the effect of node failures on the robustness of networks is studied by percolation theory both in single networks [2] and interdependent networks that interact with each other [3]. However, links frequently fail in various real-world networks, such as the failures of transmission lines in electrical power networks, path congestions in transportation networks. The concept of a line graph, that transforms links of the original graph into nodes in the line graph, can be used to understand the influence of link failures on infrastructure networks.

An undirected graph with N nodes and L links can be denoted as $G(N, L)$. The line graph $l(G)$ of a graph G is a graph in which every node in $l(G)$ corresponds to a link in G and two nodes in $l(G)$ are adjacent if and only if the corresponding links in G have a node in common [4]. The graph G is called the original graph of $l(G)$.

Line graphs are applied in various complex networks. Krawczyk *et al.* [5] propose the line graph as a model of social networks that are constructed on groups such as families, communities and school classes. Line graphs can also represent protein interaction networks where each node represents an interaction between two proteins and each link represents pairs of interaction connected by a common protein [6]. By the line graph transformation, methodologies for nodes can be extended to solve problems related to links in a graph. For instance, the link chromatic number of a graph can be computed from the node chromatic number of its line graph [7]. An Eulerian path (that can be computed rather easily in polynomial time) in a graph transforms to a Hamiltonian path (which is difficult to compute, in fact, the problem is NP-hard) in the line graph. Evan *et al.* [8] use algorithms that produce a node partition in the line graph to achieve a link partition in order to uncover

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overlapping communities of a network. Wierman *et al.* [9] improve the bond (link) percolation threshold of a graph by investigating site (node) percolation in its line graph.

Previous studies focus on various mathematical properties of line graphs. Whitney's Theorem [10] states that, if line graphs of two connected graphs G_1 and G_2 are isomorphic, the graphs G_1 and G_2 are isomorphic unless one is the complete graph K_3 and the other one is the star $K_{1,3}$. Krausz [11], Van Rooij and Wilf [12] have investigated the conditions for a graph to be a line graph. Van Rooij and Wilf [12] have studied the properties of graphs obtained by iterative usage of the line graph transformation, e.g., the line graph $l(G)$ of a graph G, the line graph $l(l(G))$ of the line graph $l(G)$, etc. Furthermore, Harary [13] has shown that for connected graphs that are not path graphs, all sufficiently high numbers of iterations of the line graph transformation produce Hamiltonian graphs[1]. The generation of a random line graph is studied in [14]. An original graph can be reconstructed [15, 16, 17] from its line graph with a computational complexity that is linear in the number of nodes N.

In this paper, we analytically study the degree distribution and the assortativity of line graphs and the relation to the degree distribution and the assortativity of their original networks. We show that the degree distribution in the line graph of the Erdős-Rényi graph follows the same pattern as the degree distribution in Erdős-Rényi. However, the line graph of an Erdős-Rényi graph is not an Erdős-Rényi graph. Additionally, we investigate the assortativity of line graphs and show that the assortativity is not linearly related to the assortativity in the original graphs. The line graphs are assortative in most cases, yet line graphs are not always assortative. We investigate graphs with negative assortativity in their line graphs. The remainder of this paper is organized as follows. The degree distribution of line graphs is presented in Section 2. Section 3 provides the assortativity of line graphs. We conclude in Section 4.

2. Degree Distribution

Random graphs are developed as models of real-world networks of several applications, such as peer-to-peer networks, the Internet and the World Wide Web. The degree distribution of Erdős-Rényi random graphs and scale free graphs are recognized by the binomial distribution and the power law distribution, respectively. This section studies the degree distribution of the line graphs of Erdős-Rényi and scale free graphs.

Let $G(N, L)$ be an undirected graph with N nodes and L links. The adjacency matrix A of a graph G is an $N \times N$ symmetric matrix with elements a_{ij} that are either 1 or 0 depending on whether there is a link between nodes i and j or not. The degree d_i of a node i is defined as $d_i = \sum_{k=1}^{N} a_{ik}$. The degree vector $d = (d_1\ d_2\ \cdots\ d_N)$ has a vector presentation as $Au = d$, where $u = (1, 1, \cdots, 1)$ is the all-one vector. The adjacency matrix [4] of the line graph $l(G)$ is $A_{l(G)} = R^T R - 2I$, where R is an $N \times L$ unsigned incidence matrix with $R_{il} = R_{jl} = 1$ if there is a link l between nodes i and j, elsewhere 0 and I is the identity matrix. The degree vector $d_{l(G)}$ of the line graph $l(G)$ is $d_{l(G)} = A_{l(G)} u_{L\times 1}$. For an arbitrary node l in the line graph $l(G)$, which corresponds to a link l connecting nodes i and j in graph G (as shown in Figure 1), the degree d_l of the node l follows

$$d_l = d_i + d_j - 2 \tag{1}$$

The random variable D_i denotes the degree of a randomly chosen node i in Erdős-Rényi graphs $G_p(N)$ and (1) shows that the degree D_l of a link l with end node i in the corresponding line graph is $D_l = D_i + D_j - 2$.

Theorem 1. *The degree distribution of the line graph $l(G_p(N))$ of an Erdős-Rényi graph $G_p(N)$ follows a binomial distribution*

$$Pr[D_l = k] = \binom{2N-4}{k} p^k (1-p)^{(2N-4-k)} \tag{2}$$

[1] A Hamiltonian graph is a graph possessing a Hamiltonian cycle which is a closed path through a graph that visits each node exactly once.

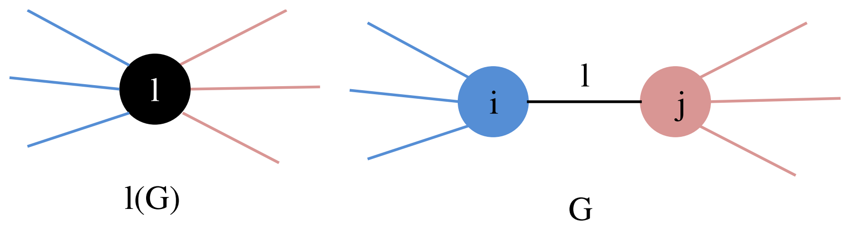


Figure 1: Node l in line Graph $l(G)$ corresponds to the link l in G.

with average degree $E[D_{l(G_p(N))}] = (2N-4)p$.

Proof. Applying (1), the degree distribution D_l of a node l in a line graph is

$$\Pr[D_l = k] = \Pr[D_i + D_j - 2 = k]$$

Using the law of total probability [18] yields

$$\Pr[D_l = k] = \sum_{m=1}^{k} \Pr[D_j = k - m + 2 \mid D_i = m] \Pr[D_i = m]$$

Since the random variables D_i and D_j in $G_p(N)$ are independent, we have

$$\Pr[D_l = k] = \sum_{m=1}^{k} \Pr[D_j = k - m + 2] \Pr[D_i = m] \tag{3}$$

An arbitrarily chosen (i.e. uniformly at random) node l in the line graph $l(G)$ corresponds to an arbitrarily chosen link in G. The degree distribution [18] of the end node i of an arbitrarily chosen link in G is

$$\Pr[D_i = m] = \frac{m\Pr[D = m]}{E[D]} \tag{4}$$

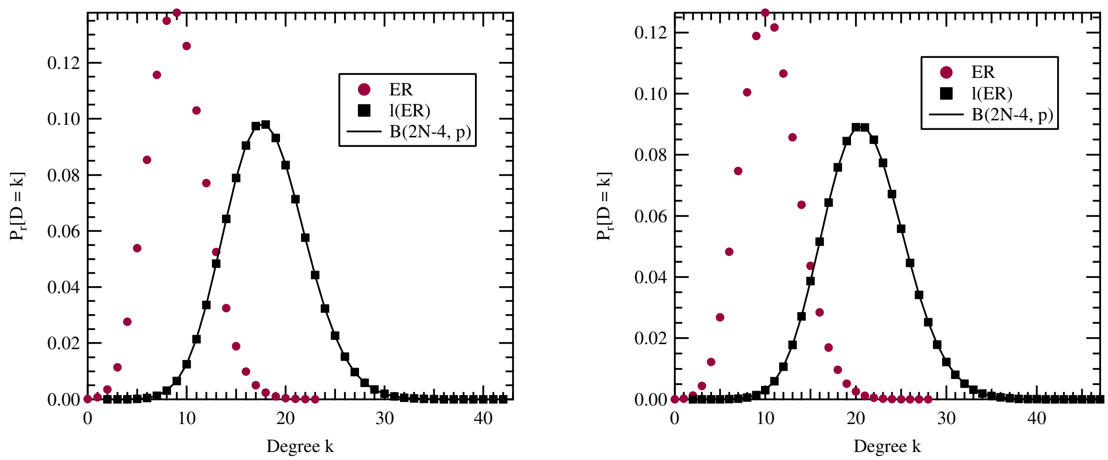
where $\Pr[D = m]$ is the degree distribution of an arbitrarily chosen node in graph G and $E[D]$ is the average degree of an arbitrarily chosen node. In an Erdős-Rényi graph, we have $\Pr[D = m] = \binom{N-1}{m} p^m (1-p)^{N-1-m}$ and $E[D] = (N-1)p$. By substituting (4) into (3) and applying the binomial distribution of random variables D_i and D_j, we have

$$\begin{aligned}
\Pr[D_l = k] &= \sum_{m=1}^{k} \frac{(k-m+2)\Pr[D = k-m+2]}{E[D]} \frac{m\Pr[D = m]}{E[D]} \\
&= \sum_{m=1}^{k} \frac{(k-m+2)\binom{N-1}{k-m-2} p^{k-m+2} (1-p)^{N-1-(k-m+2)}}{(N-1)p} \frac{m\binom{N-1}{m} p^m (1-p)^{N-1-m}}{(N-1)p} \\
&= p^k (1-p)^{2N-4-k} \sum_{m=0}^{k} \binom{N-2}{k-m} \binom{N-2}{m}
\end{aligned}$$

Using Vandermonde's identity $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$, we arrive at (2). $\square$

Theorem 1 illustrates that the degree distribution of the line graph $l(G)$ of an Erdős-Rényi graph G follows a binomial distribution with average degree $E[D_{l(G)}] = (2N-4)p$. Compared to the average degree $E[D] = (N-1)p$, the average degree of the line graph of the Erdős-Rényi graph is two times the average degree $E[D]$ of the Erdős-Rényi graph minus $2p$.

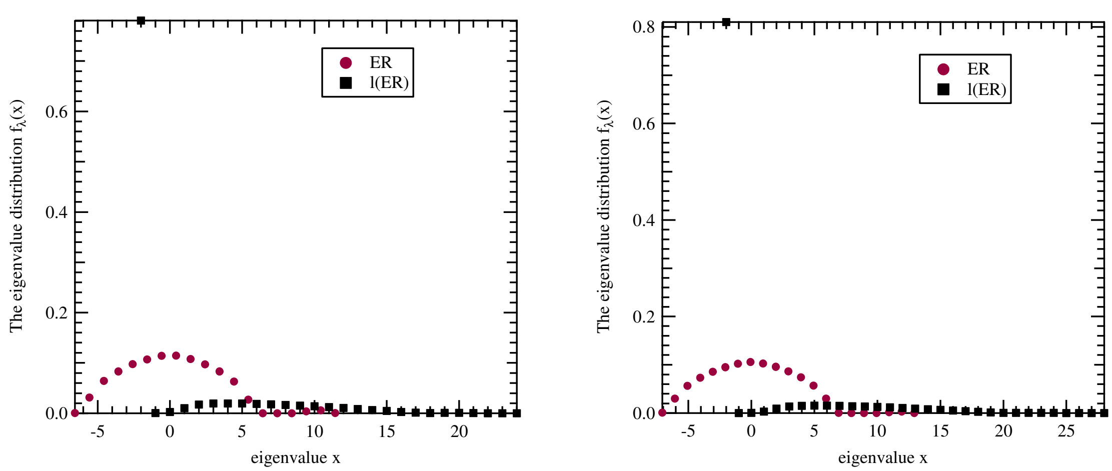
Figure 2 shows the degree distribution of the line graphs of Erdős-Rényi graphs $G_N(p)$ for $N = 100$, 200 and $p = 2p_c$ ($p_c \approx \frac{\ln N}{N}$), where 10^5 Erdős-Rényi graphs are generated. In Figures 2(a) and (b), the degree distributions of Erdős-Rényi graphs (red circle) follow a binomial distribution. The degree distribution of the corresponding line graph (black square) is fitted by a binomial distribution $B(2N-4, p)$. The simulation results agree with Theorem 1. Moreover, the average degree $E[D_{l(G)}]$ of the line graph is approximately two times the average degree $E[D]$ of the graph G.



(a) $N = 100$, $p = 2p_c$ (b) $N = 200$, $p = 2p_c$

Figure 2: The degree distribution of Erdős-Rényi graphs and their corresponding line graphs.

Since the degree distribution of the line graphs of Erdős-Rényi graphs follows a binomial distribution, we pose the question: Is the line graph of an Erdős-Rényi graph also an Erdős-Rényi graph? In order to answer this question, we investigate the eigenvalue distribution of the line graph. Figure 3 shows the eigenvalue distribution of Erdős-Rényi graphs and their line graphs. As shown in [4], the eigenvalue distribution of Erdős-Rényi graphs follows a semicircle distribution. The eigenvalue distribution of the line graphs of Erdős-Rényi graphs follows a different distribution than a semicircle distribution. Since the spectrum of a graph can be regarded as the unique fingerprint of that graph to a good approximation [19], we conclude that the line graphs of Erdős-Rényi graphs are not Erdős-Rényi graphs.



(a) $N = 100$, $p = 2p_c$ (b) $N = 200$, $p = 2p_c$

Figure 3: The eigenvalue distribution of Erdős-Rényi graphs and their corresponding line graphs. The simulations are performed on 10^5 instances.

Generating functions are powerful to study the degree distribution of networks [18]. Assuming the degree independence of nodes in graph G, Theorem 2 shows the generating function for the line graph $l(G)$ of an arbitrary graph.

Theorem 2. *Assuming that the degrees of nodes in a graph G are independent, the generating function for the degree D_l in the line graph $l(G)$ follows*

$$\varphi_{D_l}(z) = \left(\frac{E[z^{D_{l+}}]}{z}\right)^2 \tag{5}$$

where D_{l+} is the degree of the end node of an arbitrarily chosen link l in G.

Proof. The probability generating function for the degree D_l of a node l in the line graph is

$$\varphi_{D_l}(z) = E[z^{D_l}]$$

Using (1), we have

$$\varphi_{D_l}(z) = E[z^{D_i+D_j-2}]$$

Since the condition in the theorem assumes that the random variables D_i and D_j are independent and identically distributed as D_{l+}, we establish Theorem 2. □

We apply the generating function (5) in the line graph of scale-free graphs whose degree distribution follows a power law distribution with the exponent α. In Appendix Appendix A, we deduce that, for large N,

$$\Pr[D_l = k] \propto \left(\frac{1}{k+2}\right)^{\alpha^{l(G)}} \tag{6}$$

where $\alpha^{l(G)} = 2$, whereas in the original graph $\alpha^G = 3$. Equation (6) illustrates that, when we assume that the degrees in the original graph are independent, the degree distribution in the line graph of Barabási-Albert graph follows a power law degree distribution. However, due to the preferential attachment in scale-free graphs and $2L = \sum_{i=1}^{N} d_i$, the node degrees are dependent rather than independent. Correspondingly, a gap is observed in Figure 4 between the approximation equation (6) (blue circle) and the simulation result (red square).

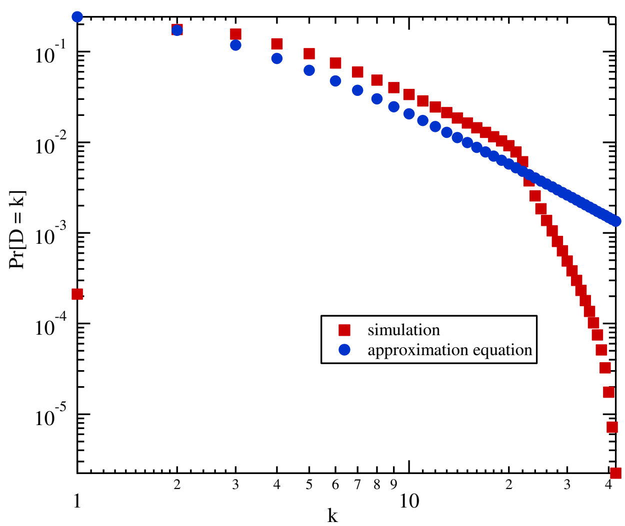


Figure 4: The degree distribution in the line graph of the Barabási-Albert graph both from simulations and the approximation equation (6). Both the x-axis and the y-axis are in log scale. The simulations are performed on 10^5 Barabási-Albert graphs with $N = 500$ and average degree 4. The cut-off in the simulation is due to the finite size of the Barabási-Albert graph.

The dependency assumption in (5) can be assessed by the total variation distance $d_{TV}(X,Y)$, defined as [18]:

$$d_{TV}(X,Y) = \sum_{k=-\infty}^{\infty} \left|\Pr[X=k] - \Pr[Y=k]\right|$$

where $\Pr[X = k]$ denotes the probability density function for (6) and $\Pr[Y = k]$ for simulations.

Figure 5 shows the total variation distance when the number of nodes N in Barabási-Albert graphs increases from 500 to 1000 with average degree 4. For each size of the original graph, 10^5 graphs are generated. Figure 5 demonstrates that $d_{TV}(X,Y)$ decreases with the number of nodes N, starting from 0.667 when $N = 500$ to 0.640 when $N = 1000$. Accordingly, the accuracy of the approximation equation (6) increases with the size of the original graph.

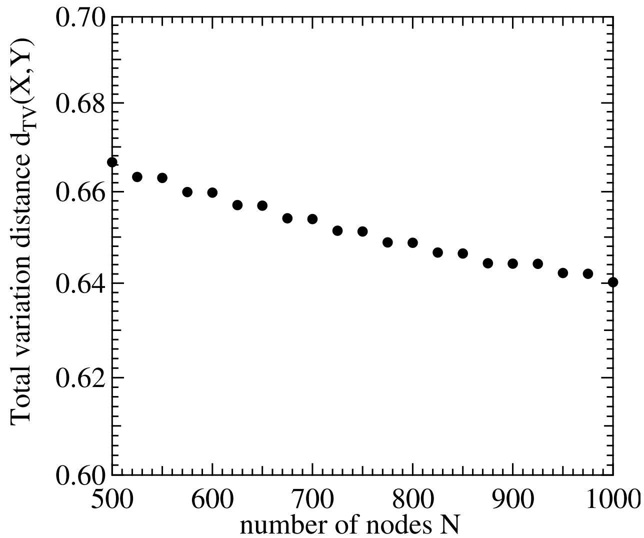


Figure 5: The total variation distance $d_{TV}(X,Y)$ when the original graph has different number of nodes from 500 to 1000.

3. Assortativity

Networks with a same degree distribution may have significantly different topological properties [20]. Networks, where nodes preferentially connect to nodes with (dis)similar property, are called (dis)assortative [21]. An overview of the assortativity in complex networks is given in [22]. Assortativity is quantified by the linear degree correlation coefficient defined as

$$\rho_{D_{l(G)}} = \frac{E[D_{l^+}D_{l^-}] - E[D_{l^+}]E[D_{l^-}]}{\sigma_{D_{l^+}}\sigma_{D_{l^-}}} \tag{7}$$

where $E[X]$ and σ_X are the mean and standard deviation of the random variable X. The definition (7) has been transformed into a graph formulation in [20]. In this section, we investigate the assortativity $\rho_{D_{l(G)}}$ of the line graph $l(G)$ and its relation to the assortativity of the graph G.

3.1. Assortativity in the line graph

In this subsection, we derive a formula for the assortativity in a general line graph, represented in Theorem 3. The relation between the assortativity in the line graph and the assortativity in the original graph is shown in Corollary 1.

Theorem 3. *The assortativity in the line graph $l(G)$ of a general graph G is*

$$\rho_{D_{l(G)}} = 1 - \frac{d^T A \Delta d - N_4}{3d^T A \Delta d + \sum_{k=1}^N d_k^4 - 2\sum_{k=1}^N d_k^3 - 2N_3 - \frac{(N_3 + \sum_{k=1}^N d_k^3 - 2N_2)^2}{N_2 - N_1}}$$

where d is the degree vector, $\Delta = diag(d_i)$ is the diagonal matrix with the nodal degrees in G and $N_k = u^T A^k u$ is the total number of walks of length k.

The proof for Theorem 3 is given in Appendix Appendix B. In order to investigate the relation between the assortativity of the line graph $l(G)$ and the assortativity of the graph G, Corollary 1 rephrases the assortativity $\rho_{D_{l(G)}}$ of the line graph $l(G)$ in terms of the assortativity ρ_D of the graph G.

Corollary 1. *The assortativity $\rho_{D_{l(G)}}$ of the line graph can be written in terms of the assortativity ρ_D of the graph G as*

$$\rho_{D_{l(G)}} = 1 - \frac{(d^T A \Delta d - N_4)\mu^2}{(N_2 - N_1)\left(-4(1+\rho_D)^2\left(\frac{1}{N_1}\sum_{i=1}^N d_i^3 - \left(\frac{N_2}{N_1}\right)^2\right)^2 + 2\mu^2(1+\rho_D)\left(\frac{1}{N_1}\sum_{i=1}^N d_i^3 - \left(\frac{N_2}{N_1}\right)^2\right) + \mu u_3\right)}$$

where $\mu = E[D_{l(G)}]$ and $u_3 = E[(D_{l(G)} - E[D_{l(G)}])^3]$.

The proof for Corollary 1 is given in Appendix Appendix C. Corollary 1 indicates that the assortativity of the line graph is not linearly related to the assortativity of the original graph. For the Erdős-Rényi graphs, a relatively precise relation between the assortativity of the line graph and the one of the original graph is given in Theorem 4.

Theorem 4. *The difference between the assortativity $\rho_{D_{l(G)}}$ of the line graph of an Erdős-Rényi graph $G_N(p)$ and the assortativity ρ_{D_G} of $G_N(p)$ converges to 0.5 in the limit of large graph size N.*

Proof. Based on the definition in equation (7) and denoting $l^+ = i \sim c$ and $l^- = c \sim j$, we have

$$\begin{aligned}\rho_{D_{l(G)}} &= \frac{E[(D_i + D_c)(D_j + D_c)] - E[D_i + D_c]E[D_j + D_c]}{\sigma_{D_i+D_c}\sigma_{D_j+D_c}} \\ &= \frac{E[D_iD_j] - E[D_i]E[D_j] + E[D_iD_c] - E[D_i]E[D_c] + E[D_jD_c] - E[D_j]E[D_c] + E[D_c^2] - E^2[D_c]}{Var[D_i] + Var[D_c] + 2E[(D_i - E[D_i])(D_c - E[D_c])]}\end{aligned}$$

In the connected Erdős-Rényi random graph in the limit of large graph size N, the assortativity ρ_{D_G} converges to zero [4] and we have

$$E[D_iD_j] - E[D_i]E[D_j] \approx 0$$

Similarly, $E[D_iD_c] - E[D_i]E[D_c] \approx 0$ and $E[D_jD_c] - E[D_j]E[D_c] \approx 0$. Combining with $E[(D_i - E[D_i])(D_c - E[D_c])] = E[D_iD_c] - E[D_i]E[D_c] \approx 0$, we arrive at

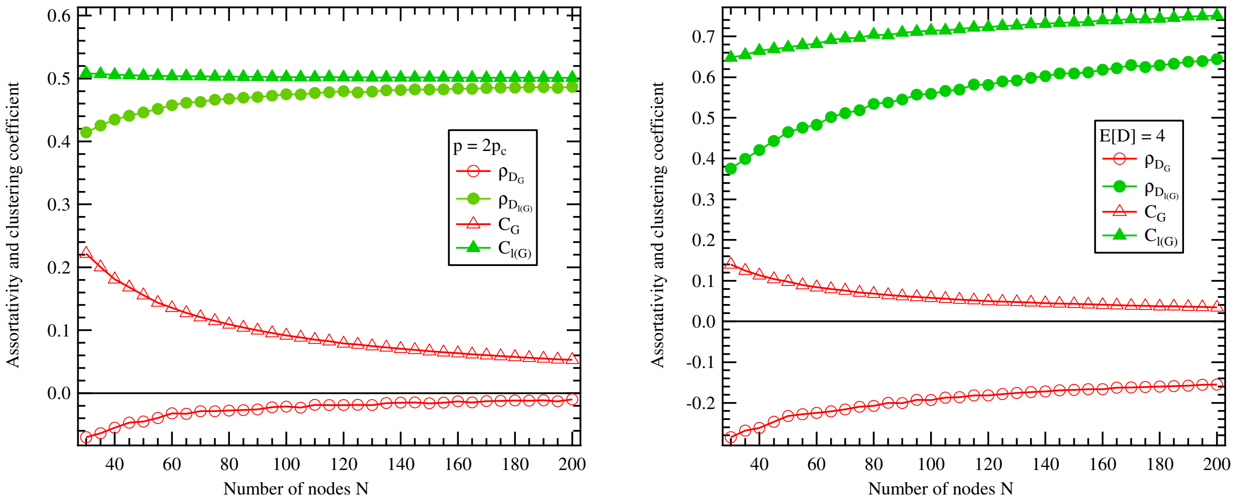
$$\rho_{D_{l(G)}} \approx \frac{E[D_c^2] - E^2[D_c]}{2\text{Var}[D_c]} = 0.5$$

□

In order to verify Theorem 4, Figure 6 shows the assortativity of (a) Erdős-Rényi graphs, (b) Barabási-Albert graphs, and the assortativity of their corresponding line graphs. In Figure 6(a), the assortativity of $G_p(N)$ converges to 0 with the increase of the graph size N. Correspondingly, the assortativity in the line graph of $G_p(N)$ converges to 0.5 which confirms Theorem 4. Based on the assortativity ρ_D of a connected Erdős-Rényi graph $G_p(N)$, which is zero [4, 21] in the limit of large graph size, we again verify that the line graph of an Erdős-Rényi graph is not an Erdős-Rényi graph. Figure 6(b) illustrates the assortativity $\rho_{D_{l(G)}}$ of the line graph of the Barabási-Albert graph is also positive and increases with the graph size.

Youssef *et al.* [23] show that the assortativity is related to the clustering coefficient[2] C_G. Specifically, assortative graphs tend to have a higher number $\blacktriangle_G$ of triangles and thus a higher clustering coefficient compared to disassortative graphs. Figure 6 shows that the assortative line graphs of both Erdős-Rényi and Barabási-Albert graph have a higher clustering coefficient (above 0.5). The results agree with the findings in [23].

[2]The clustering coefficient $C_G = \frac{3\blacktriangle_G}{N_2}$ is defined as three times the number $\blacktriangle_G$ of triangles divided by the number N_2 of connected triples.



(a) Erdős-Rényi graph.

(b) Barabási-Albert graph.

Figure 6: Assortativity ρ_D and clustering coefficient C_G of the (a) Erdős-Rényi graph $G_p(N)$ with $p = 2p_c$, (b) Barabási-Albert graph with the average degree $E[D] = 4$ and the corresponding line graph $l(G)$.

Table 1: Assortativity of real-world networks and their corresponding line graphs.

Networks	Nodes	Links	ρ_D	$\rho_{D_{l(G)}}$
Co-authorship Network [24]	379	914	−0.0819	0.6899
US airports [25]	500	2980	−0.2679	0.3438
Dutch Soccer [26]	685	10310	−0.0634	0.5170
Citation[3]	2678	10368	−0.0352	0.8127
Power Grid	4941	6594	−0.0035	0.7007

Table 1 shows the assortativity of real-world networks and their corresponding line graphs. As shown in the table, the line graphs of all the listed networks show assortative mixing even though the original networks show dissortative mixing.

3.2. Negative assortativity in line graphs

Although the assortativity of a line graph is predominantly positive, we cannot conclude that the assortativity in any line graph is positive. This subsection presents graphs, whose corresponding line graphs possesses a negative assortativity.

3.2.1. The Line graph of a path graph

A path graph P_N is a tree with two nodes of degree 1, and the other $N-2$ nodes of degree 2. The line graph $l(P)$ of a path graph P_N is still a path graph but with $N-1$ nodes. Observation 1 demonstrates that the assortativity in the line graph of a path graph is always negative.

Observation 1. *The assortativity of the line graph $l(P)$ of a path P_N is*

$$\rho_{D_{l(P)}} = -\frac{1}{N-3}$$

where N is the number of nodes in the original path graph.

[3]http://vlado.fmf.uni-lj.si/pub/networks/data/

Proof. The reformulation [4] of the assortativity can be written as

$$\rho_D = 1 - \frac{\sum_{i\sim j}(d_i - d_j)^2}{\sum_{i=1}^{N-1}(d_i)^3 - \frac{1}{2L}(\sum_{i=1}^{N-1} d_i^2)^2} \tag{8}$$

Since the line graph of a path with N nodes is a path graph with $N-1$ nodes, where 2 nodes have node degree 1 and the other $(N-1)-2$ nodes have degree 2, we have that

$$\sum_{i=1}^{N-1} d_i^k = 2 \times 1^k + ((N-1)-2) \times 2^k \tag{9}$$

and

$$\sum_{i\sim j}(d_i - d_j)^2 = 2 \times 1^2 \tag{10}$$

Applying equations (9) and (10) into (8), we establish the Observation 1. □

The negative assortativity $\rho_{D_{l(P)}}$ of the line graph $l(P)$ of a path graph is an exception to the positive assortativity of the line graphs of the Erdős-Rényi graph, Barabási-Albert graph and real-world networks given in Table 1. Moreover, the assortativity of the line graph $l(P)$ is a fingerprint for the line graph $l(P)$ to be a path graph.

3.2.2. The Line graph of a path-like graph

Let $P^{m_1, m_2, \cdots, m_t}_{n_1, n_2, \cdots, n_t, p}$ be a path of p nodes ($1 \sim 2 \sim \cdots \sim p$) with pendant paths of n_i links at nodes m_i, following the definition in [27]. We define the graph D_N through $D_N = P^2_{1,\ N-1}$ as drawn in Fig. 7. Observation 2 shows that the assortativity in the corresponding line graph $l(D_N)$ is always negative.

Figure 7: The graph D_N whose line graph has the negative assortativity.

Observation 2. *The assortativity of the line graph $l(D_N)$ of the graph D_N in Figure 7 is*

$$\rho_{D_{l(D_N)}} = -\frac{1}{2N-3}$$

where N is the number of nodes in the graph D_N.

Proof. Since 1 node has node degree 1, 1 node has node degree 3 and the other $(N-1)-2$ nodes have degree 2, we have that

$$\sum_{i=1}^{N-1} d_i^k = 1 \times 1^k + 1 \times 3^k + ((N-1)-2) \times 2^k \tag{11}$$

and

$$\sum_{i\sim j}(d_i - d_j)^2 = 1 \times 1^2 + 3 \times 1^2 \tag{12}$$

Applying equations (11) and (12) into (8), we establish the Observation 2. □

Figure 8: The graph E_N whose line graph has the negative assortativity.

We define the graph E_N through $E_N = P^3_{1,\ N-1}$ as drawn in Fig. 8. The graph E_N is obtained from D_N by moving the pendant path from node 2 to node 3. The assortativity of the line graph $l(E_N)$ of the graph E_N is

$$\rho_{D_{l(E_N)}} = -\frac{1}{N-2}$$

For the graphs $P^{m_i}_{1,\ N-1}$ with one pendant path of 1 link at node m_i ($i = 2,\ 3,\ \cdots,\ N-2$), there are $N-3$ positions to attach the pendant path. Since the position for adding the pendant path is symmetric at $\lceil\frac{N-1}{2}\rceil$. We only consider i from 2 to $\lceil\frac{N-1}{2}\rceil$. Among all the graphs $P^{m_i}_{1,\ N-1}$ where $i = 2,\ 3,\ \cdots,\ \lceil\frac{N-1}{2}\rceil$), the line graphs of the graph D_N and E_N always have negative assortativity. The line graph of the graph $P^{m_i}_{1,\ N-1}$, where $i = 4,\ 5,\ \cdots,\ \lceil\frac{N-1}{2}\rceil$, has negative assortativity when the size N is small and has positive assortativity as N increases.

The graph $\widetilde{D}_N$ is defined through $\widetilde{D}_N = P^{2,\ N-3}_{1,\ 1,\ N-2}$ as drawn in Fig. 9. Observation 3 shows that the assortativity in the corresponding line graph $l(\widetilde{D}_N)$ is always negative.

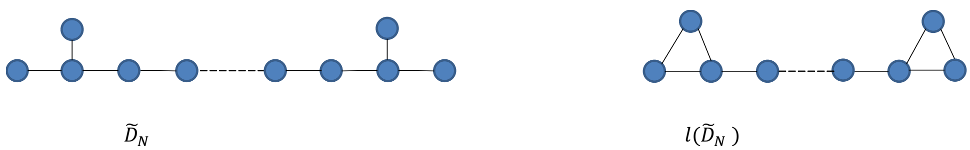


Figure 9: The graph $\widetilde{D}_N$ whose line graph has the negative assortativity.

Observation 3. *The assortativity of the line graph $l(\widetilde{D}_N)$ of the graph $\widetilde{D}_N$ in Figure 9 is*

$$\rho_{D_{l(\widetilde{D}_N)}} = -\frac{3}{N-3}$$

where N is the number of nodes in $\widetilde{D}_N$.

Proof. Since 2 nodes have node degree 3 and the other $(N-1)-2$ nodes have degree 2, we have that

$$\sum_{i=1}^{N-1} d_i^k = 2 \times 3^k + ((N-1)-2) \times 2^k \tag{13}$$

and

$$\sum_{i\sim j}(d_i - d_j)^2 = 6 \times 1^2 \tag{14}$$

Applying equations (13) and (14) into (8), we establish the Observation 3. □

The graphs $\widetilde{E}_N$ and $\widetilde{F}_N$ are defined through $\widetilde{E}_N = P^{2,\ N-4}_{1,\ 1,\ N-2}$ and $\widetilde{F}_N = P^{3,\ N-4}_{1,\ 1,\ N-2}$ as drawn in Fig. 10. The assortativity for the line graph of $\widetilde{E}_N$ is

$$\rho_{D_{l(\widetilde{E}_N)}} = -\frac{16}{5N-16}$$

The assortativity for the line graph of $\widetilde{F}_N$ is

$$\rho_{D_{l(\widetilde{F}_N)}} = -\frac{25}{7N-25}$$

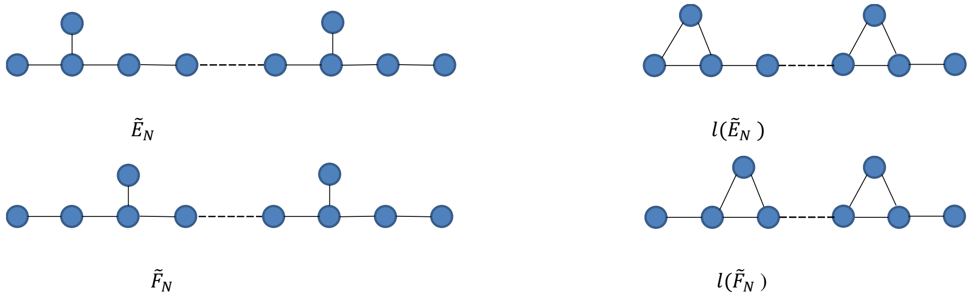


Figure 10: The graphs $\widetilde{E}_N$ and $\widetilde{F}_N$ whose line graphs have the negative assortativity.

Graphs $\widetilde{D}_N$, $\widetilde{E}_N$, $\widetilde{F}_N$ are the graphs whose line graphs always have the negative assortativity. For the remaining graphs $P^{m_i,\ m_j}_{1,\ 1,\ N-2}$, $i \neq j$, their line graphs have negative assortativity when N is small. As N increases, the assortativity of the line graphs is positive.

3.2.3. Line graph of non-trees

Both the path graphs and path-like graphs are trees. In this subsection, we study whether there exist non-trees whose line graphs have negative assortativity.

We start by studying the non-trees $l(D_N)$, $l(E_N)$ and $l(\widetilde{D}_N)$, $l(\widetilde{E}_N)$, $l(\widetilde{F}_N)$ in Figures 7-10. The non-tree graphs consist of cycles of 3 nodes connected by disjoint paths. The line graph of the non-tree $l(D_N)$ is denoted as $l(l(D_N))$, which is also the line graph of the line graph of D_N. By simulations we determine the non-tree graphs whose line graphs have negative assortativity. The results are given in Figures 11 and 12.

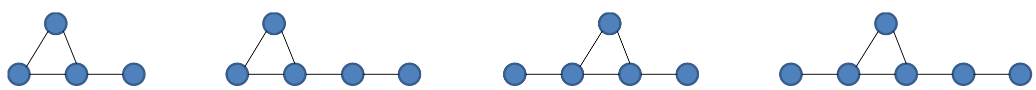


Figure 11: Non-tree graphs $l(D_N)$, $l(E_N)$ whose line graphs $l(l(D_N))$, $l(l(E_N))$ have negative assortativity.

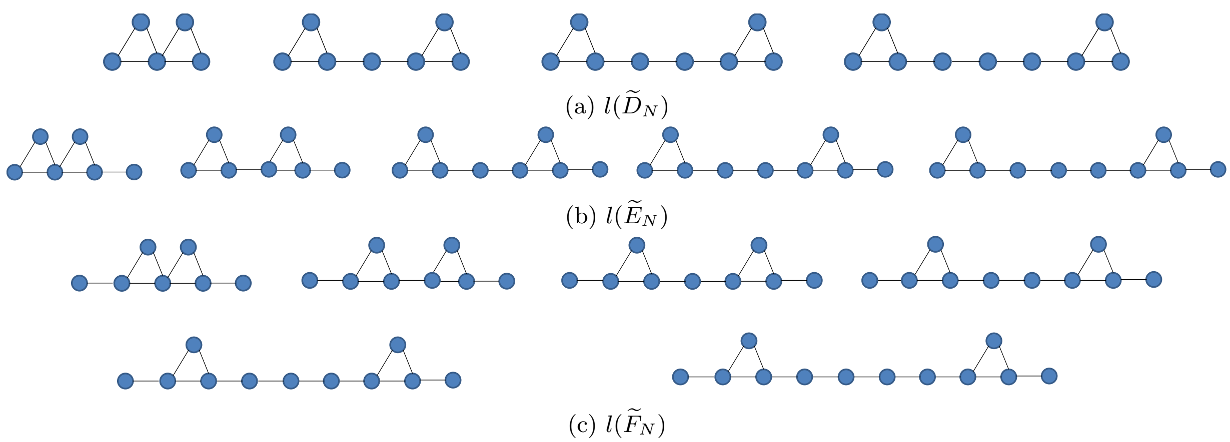


Figure 12: Non-tree graphs $l(\widetilde{D}_N)$, $l(\widetilde{E}_N)$, $l(\widetilde{F}_N)$ whose line graphs $l(l(\widetilde{D}_N))$, $l(l(\widetilde{E}_N))$, $l(l(\widetilde{F}_N))$ have negative assortativity.

As shown in Figures 11 and 12, for the line graphs of the non-trees to have negative assortativity, there can be either 1 or 2 cycles in the non-trees. In Figure 11, the line graph $l(l(E_N))$ of $l(E_N)$ has 1 cycle connected by two paths and the maximal path length is 2. In Figure 12, two cycles are connected by maximal 3 paths and the maximal path length is 4 in the line graph $l(l(\widetilde{F}_N))$. Moreover, for a line graph to have negative assortativity, the size of the original graph is in general small, less than 14 nodes in our simulations.

4. Conclusion

Topological characteristics of links influence the dynamical processes executed on complex networks triggered by links. The line graph, which transforms links from a graph to nodes in its line graph, generalizes the topological properties from nodes to links. This paper investigates the degree distribution and the assortativity of line graphs. The degree distribution of the line graph of an Erdős-Rényi random graph follows the same pattern of the degree distribution as the original graph. We derive a formula for the assortativity of the line graph. We indicate that the assortativity of the line graph is not linearly related to the assortativity of the original graph. Moreover, the assortativity is positive for the line graphs of Erdős-Rényi graphs, Barabási-Albert graphs and most real-world networks. In contrast, certain types of trees, path and path-like graphs, have negative assortativity in their line graphs. Furthermore, non-trees consisting of cycles and paths can also have negative assortativity in their line graphs.

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Appendix A. Proof of equation (6)

The degree distribution in scale free graphs G is

$$\Pr[D = k] = \frac{k^{-\alpha}}{c_1}, \quad k = s, \cdots, K \tag{A.1}$$

where $c_1 = \sum_{k=s}^{K} k^{-\alpha}$ is the normalization constant and s is the minimum degree and K is the maximum degree in G. Assuming the node degrees in the scale free graph are independent, the generating function for the line graph of scale free graphs can be written as equation (5). Substituting the derivative of the generating function $\varphi_D^{'}(z) = \frac{1}{E[D]}\sum_{k=0}^{N-1} kz^{k-1}\Pr[D=k]$ and the average degree $E[D] = \sum_{k=0}^{N-1} k\Pr[D=k] = \frac{c_2}{c_1}$, where $c_2 = \sum_{k=s}^{K} k^{1-\alpha}$, into equation (5) yields

$$\varphi_{D_l}(z) = \left(\frac{c_1}{c_2}\right)^2 \left(\varphi_D^{'}(z)\right)^2 \tag{A.2}$$

and the Taylor coefficients obey

$$\Pr[D_l = k] = \left(\frac{c_1}{c_2}\right)^2 \frac{1}{k!} \frac{d^k\left(\varphi_D^{'}(z)\right)^2}{dz^k}\Bigg|_{z=0}$$

Using the Leibniz's rule $(fg)^{(k)} = \sum_{m=0}^{k}\binom{k}{m} f^{(m)}g^{(k-m)}$, where $f = g = \varphi_D^{'}(z)$, yields

$$\Pr[D_l = k] = \left(\frac{c_1}{c_2}\right)^2 \frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}\frac{d^{m+1}(\varphi_D(z))}{dz^{m+1}}\frac{d^{k-m+1}(\varphi_D(z))}{dz^{k-m+1}}\Bigg|_{z=0}$$

Substituting $k!\Pr[D=k] = \frac{d^k(\varphi_D(z))}{dz^k}\Big|_{z=0}$, we arrive at

$$\Pr[D_l = k] = \frac{1}{k!}\left(\frac{c_1}{c_2}\right)^2 \sum_{m=0}^{k}\frac{k!}{m!(k-m)!}(m+1)!\Pr[D=m+1](k-m+1)!\Pr[D=k-m+1]$$

Applying the power law degree distribution in equation (A.1), we have

$$\Pr[D_l = k] = \frac{1}{c_2^2}\sum_{m=1}^{k+1}\Big(m(k+2-m)\Big)^{1-\alpha} \tag{A.3}$$

For $\alpha = 3$, we transform equation (A.3) in the following form:

$$c_2^2\Pr[D_l = k] = \frac{1}{(k+2)^3}\sum_{i=1}^{k+1}\frac{1}{(\frac{i}{k+2})^2(1-\frac{i}{k+2})^2}\frac{1}{k+2} \tag{A.4}$$

We use the following expression between a sum in the limit to infinity and a definite integral [28]

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$$

We set $\Delta x = \frac{1}{k+2}$, $x_i = i\Delta x = \frac{i}{k+2}$, $f(x) = \frac{1}{x^2(1-x)^2}$ and (A.4) boils down to

$$c_2^2\Pr[D_l = k] = \frac{1}{(k+2)^3}\sum_{i=1}^{k+1} f(x_i)\Delta x \tag{A.5}$$

We consider the case of limit to infinity for k ($k \to \infty$) or k very large and evaluate the sum $\sum_{i=1}^{k+1} f(x_i)\Delta x$, which can be transformed into

$$\sum_{i=1}^{k+1} f(x_i)\Delta x \approx \int_{\frac{1}{k+2}}^{\frac{k+1}{k+2}} f(x)dx \tag{A.6}$$

Now,

$$\int_{\frac{1}{k+2}}^{\frac{k+1}{k+2}} f(x)dx = \int_{\frac{1}{k+2}}^{\frac{k+1}{k+2}} \frac{1}{x^2(1-x)^2}dx$$
$$= \int_{\frac{1}{k+2}}^{\frac{k+1}{k+2}} (\frac{2}{x} + \frac{2}{1-x} + \frac{1}{x^2} + \frac{1}{(1-x)^2})dx$$
$$= 2\big(2\ln(k+1) + \frac{k(k+2)}{k+1}\big) \tag{A.7}$$

Using (A.7) and (A.6) into (A.5), leads to

$$c_2^2 \Pr[D_l = k] \approx \frac{2}{(k+2)^2}(\frac{2\ln(k+1)}{k+2} + \frac{k}{k+1})$$

Since $\lim_{k\to\infty} \frac{\ln(k+1)}{k+2} = 0$ and $\lim_{k\to\infty} \frac{k}{k+1} = 1$, we arrive at

$$\Pr[D_l = k] \approx \frac{1}{c_2^2}(k+2)^{-2} \tag{A.8}$$

Appendix B. Proof for Theorem 3

Proof. A link l with end nodes l^+ and l^- in the line graph $l(G)$ corresponds to a connected triplet in G. Without loss of generality, we assume that nodes l^+ and l^- in the line graph correspond to links $l^+ = i \sim c$ and $l^- = j \sim c$, where links $i \sim c$ and $j \sim c$ share a common node c, in the original graph as shown in Figure B.13. The degree in line graph is $d_{l^+} = d_i + d_c - 2$ and $d_{l^-} = d_j + d_c - 2$. Since subtracting 2 everywhere does not

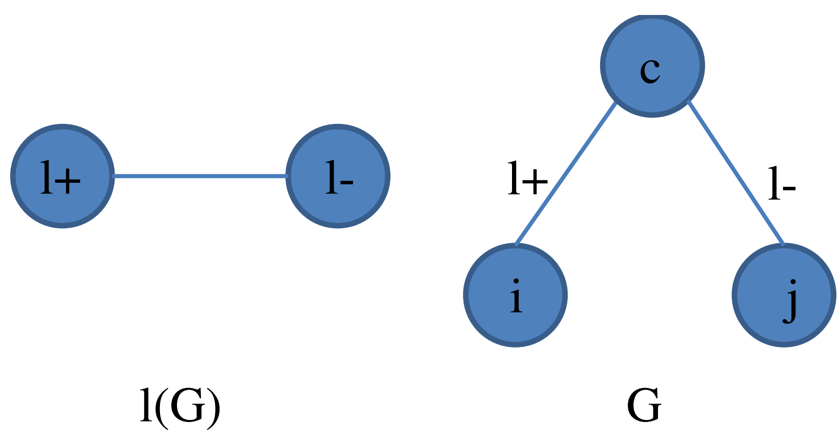

Figure B.13: Link transformation.

change the linear correlation coefficient, we proceed with $d_{l^+} = d_i + d_c$ and $d_{l^-} = d_j + d_c$. First, we compute the joint expectation

$$E[D_{l^+}D_{l^-}] = \frac{\sum\limits_{i=1}^{N}\sum\limits_{\substack{j=1\\j\neq i}}^{N}\sum\limits_{c=1}^{N}(d_i+d_c)(d_j+d_c)a_{ic}a_{jc}}{2L_{l(G)}}$$

$$= \frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}(d_i+d_c)(d_j+d_c)a_{ic}a_{jc} - \sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}(d_i+d_c)^2a_{ic}^2}{2L_{l(G)}}$$

$$= \frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}a_{jc}d_j + 2\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}a_{jc}d_c + \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_c^2a_{ic}a_{jc}}{2L_{l(G)}}$$

$$- \frac{2\sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}d_i^2a_{ic}^2 + 2\sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}^2d_c}{2L_{l(G)}}$$

With $\sum_{j=1}^{N}a_{jc} = d_c$, we arrive at

$$E[D_{l^+}D_{l^-}] = \frac{d^TA^2d + 2d^TA\Delta d + \sum_{i=c}^{N}d_c^4 - 2\sum_{i=1}^{N}d_i^3 - 2d^TAd}{2L_{l(G)}} \tag{B.1}$$

The average degree $E[D_{l^+}] = E[D_i + D_c]$ is the average degree of two connected nodes i and c from a triplet (see Figure B.13) in the original graph. Thus,

$$E[D_{l^+}] = \frac{\sum\limits_{i=1}^{N}\sum\limits_{\substack{j=1\\j\neq i}}^{N}\sum\limits_{c=1}^{N}(d_i+d_c)a_{ic}a_{jc}}{2L_{l(G)}}$$

$$= \frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}a_{jc} + \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_ca_{ic}a_{jc} - \sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}^2 - \sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}d_ca_{ic}^2}{2L_{l(G)}}$$

from which

$$E[D_{l^+}] = \frac{d^TAd + \sum_{c=1}^{N}d_c^3 - 2d^Td}{2L_{l(G)}} \tag{B.2}$$

The variance $\sigma_{D_{l^+}}^2 = Var[D_{l^+}] = E[D_{l^+}^2] - (E[D_{l^+}])^2$ and

$$E[D_{l^+}^2] = \frac{\sum\limits_{i=1}^{N}\sum\limits_{\substack{j=1\\j\neq i}}^{N}\sum\limits_{c=1}^{N}(d_i+d_c)^2a_{ic}a_{jc}}{2L_{l(G)}}$$

$$= \frac{\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_i^2a_{ic}a_{jc} + 2\sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}a_{jc}d_c + \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N}\sum\limits_{c=1}^{N}d_c^2a_{ic}a_{jc}}{2L_{l(G)}}$$

$$- \frac{2\sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}d_i^2a_{ic}^2 + 2\sum\limits_{i=1}^{N}\sum\limits_{c=1}^{N}d_ia_{ic}^2d_c}{2L_{l(G)}}$$

which we rewrite as

$$E[D_{l^+}^2] = \frac{3d^TA\Delta d + \sum_{c=1}^{N}d_c^4 - 2\sum_{i=1}^{N}d_i^3 - 2d^TAd}{2L_{l(G)}} \tag{B.3}$$

The number of links $L_{l(G)}$ in a line graph is [4]

$$L_{l(G)} = \frac{1}{2}d^T d - L = \frac{1}{2}(N_2 - N_1) \tag{B.4}$$

After substituting equations (B.1-B.4) into (7), we establish the Theorem. □

Appendix C. Proof for Corollary 1

Proof. Using the variance $\sigma^2_{D_{l^+}} = \text{Var}[D_{l^+}] = E[D^2_{l^+}] - (E[D_{l^+}])^2$, we rewrite the definition of assortativity (7) as

$$\rho_{D_{l(G)}} = 1 + \frac{E[D_{l^+}D_{l^-}] - E[D^2_{l^+}]}{\sigma^2_{D_{l^+}}} \tag{C.1}$$

According to equations (B.1) and (B.3), we have that

$$\begin{aligned} E[D_{l^+}D_{l^-}] - E[D^2_{l^+}] &= \frac{d^T A^2 d - d^T A \Delta d}{2L_{l(G)}} \\ &= \frac{N_4 - d^T A \Delta d}{2L_{l(G)}} \end{aligned} \tag{C.2}$$

The variance $\text{Var}[D_{l^+}]$ of the end node of an arbitrarily chosen link can be written in terms of the variance $\text{Var}[D_{l(G)}]$ of an arbitrarily chosen node [29]

$$\sigma^2_{D_{l^+}} = \frac{\mu u_3 - (\text{Var}[D_{l(G)}])^2 + \mu^2 \text{Var}[D_{l(G)}]}{\mu^2} \tag{C.3}$$

where $\mu = E[D_{l(G)}]$ and $u_3 = E[(D_{l(G)} - E[D_{l(G)}])^3]$. The variance $\text{Var}[D_{l(G)}]$ of an arbitrarily chosen node can be written in terms of the assortativity [4]

$$\text{Var}[D_{l(G)}] = 2(1 + \rho_D)\left(\frac{1}{N_1}\sum_{i=1}^{N} d_i^3 - \left(\frac{N_2}{N_1}\right)^2\right) \tag{C.4}$$

Substituting (C.2-C.4) into (C.1), we prove the Corollary 1. □