Complete game-theoretic characterization of SIS epidemics protection strategies*

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Abstract— **Defining an optimal protection strategy against viruses, spam propagation or any other kind of contamination process is an important feature for designing new networks and architectures. In this work, we consider decentralized optimal protection strategies when a virus is propagating over a network through a Susceptible Infected Susceptible (SIS) epidemic process. We assume that each node in the network can fully protect itself from infection at a constant cost, or the node can use recovery software, once it is infected.**

We model our system using a game theoretic framework. Based on this model, we find pure and mixed equilibria, and evaluate the performance of the equilibria by finding the Price of Anarchy (PoA) in several network topologies. Finally, we give numerical illustrations of our results.

I. INTRODUCTION

Virus spread processes in networks can be explained, using epidemic models [1], [2], [3], [4]. The probability of infection over time [1], [2], [5], especially in the steady-state, and the epidemic threshold [2], [6], [7] in relation to the properties of the underlying network have been widely studied in the past. We consider the Susceptible Infected Susceptible (SIS) model, which is one of the mostly studied epidemic models [1], [5]. In the SIS model, at a specific time, the state of each node is either susceptible or infected. The recovery (curing) process of each infected node is an independent Poisson process with a recovery rate δ. Each infected node infects each of its susceptible neighbors with a rate β, which is also an independent Poisson process. Protection strategies [8], [9], [10], [11], such as immunization [12], or quarantining [13] prevent nodes from being infected, while additional mechanisms [14], like anti-spyware software or clean-up tools, could clean the virus from an infected node.

This paper considers investment games that find appropriate protection strategies against SIS virus spread. In particular, we consider a game, in which, each node (host) is a player in the game and decides individually whether or not to invest in antivirus protection. Further, if a host does not invest in antivirus protection, it remains vulnerable to the virus spread process, but can recover (e.g., by a system recovery or clean-up software). The utility or payoff of each node (player) is: (i) the investment cost, if the node decides to invest in antivirus software or else (ii) the cost of being infected, which is proportional to the infection probability in the epidemic steady-state.

Game theoretical studies for networks problems have been conducted, in routing [15], network flow [16], or workload on the cloud [17], employing standard game-theoretic concepts [18], [19] such as pure Nash or mixed equilibrium. The Price of Anarchy (PoA) [20], [19] is often used as an equilibrium performance evaluation metric. While many papers have been focused on the process of virus spread and network immunizations to suppress the spread, few epidemic studies use game theory as a tool. Omić *et al.* [21] tune the strength of the nodal antivirus protection i.e. how big the curring rates (δ_i) should be taken. Contrarily to [21], (i) we fix the curing and infection rates, which are not part of the game, and the decision consists of a player's choice to invest in an antivirus or not; (ii) we also consider mixed strategies Nash Equilibrium. The goal of [21] is in finding the optimal δ_i for each player i, while this paper targets the optimal decision of taking an anti-virus that fully protects the host, because today's antivirus software packages provide accurate and up to date virus protection. The related papers on security games [22], [23], [24], [25] are usually applied in non-SIS environments (e.g., (i) without considering the infection state of the neighbors and (ii) without an additional mechanism for recovery), for generalized game settings [25] or by assigning nodal weights to reflect the security level [24].

Our main contributions are summarized as follows:

1) We prove that the game is a potential game by showing that it is equivalent to a congestion game. We determine a closed-form expression for the pure equilibrium for a *single community/full mesh* network. We also prove the existence and uniqueness of a mixed equilibrium.
2) We provide a measure of the equilibrium efficiency based on the Price of Anarchy (PoA).
3) We extend our equilibrium analysis to *bipartite* networks, where we show that multiple equilibria are possible. At an equilibrium, the number of nodes that invest in one partition is often close to the number of nodes that invest in the other partition.

The paper is organized as follows. The SIS epidemic model is introduced in Section II. Sections III and IV describe the game models in a single community (full mesh) and bipartite network, respectively and subsequently prove game theoretic results (pure, mixed equilibria and the Price of Anarchy). The conclusion is given in Section V. The technical proofs of the presented results and additional results

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for the studied and other network topologies are given online, in our extended version [26].

II. SIS epidemics on networks

We start with the general case, where the network G is connected, undirected and unweighted with N of nodes. The virus behaves as an SIS epidemic, where an infected node can infect each of its direct, healthy neighbors with rate β. Each node can be cured at rate δ, after which the node becomes healthy, but susceptible again to the virus. Both infection and curing process are independent Poisson processes. All nodes in the network G are prone to a virus that can re-infect the nodes multiple times.

We denote the viral probability of infection for node i at time t by $v_i\left(N;t\right)$. For each node i of the graph with N nodes, the SIS governing equation, under the standard N-Intertwined mean-field approximation (NIMFA) [5], is given by

$$\frac{dv_i\left(N;t\right)}{dt}=-\delta v_i\left(N;t\right)+\beta(1-v_i(N;t))\sum_{j=1}^{N}a_{ij}v_j\left(N;t\right), \tag{1}$$

where $a_{ij}=1$, if nodes i and j are directly connected by a link, while $a_{ij}=0$ if they are not. The physical interpretation of the governing differential equation (1) is the following: the infection probability of a node i changes over time by two competing processes: (i) all infected neighbors of node i try to infect him with Poisson rate β, while node i is healthy (with probability $1-v_i(t)$), and (ii) node i can be cured with a Poisson rate δ, while infected with probability $v_i(t)$.

We further confine ourselves to the stationary regime of the SIS epidemic process, meaning $\lim_{t\to\infty}\frac{dv_i(N;t)}{dt}=0$. We denote $\tau=\frac{\beta}{\delta}$ the effective spreading rate and $v_{i,\infty}(N)=\lim_{t\to\infty}v_i\left(N;t\right)$ the probability of node i being infected in the stationary regime, which is not a function of time t, but we put ∞ as an index for distinguishing the stationary regime. Now, the governing equation (1) boils down to,

$$0=-v_{i,\infty}\left(N\right)+\tau(1-v_{i,\infty}(N))\sum_{j=1}^{N}a_{ij}v_{j,\infty}\left(N\right) \tag{2}$$

Based on (2), $v_{i,\infty}\left(N\right)$ could be expressed as [1], [5],

$$v_{i,\infty}(N)=1-\frac{1}{1+\tau\sum_{j=1}^{N}a_{ij}v_{j,\infty}(N)} \tag{3}$$

for $\forall i=1,2,\ldots,N$. These steady-state equations only have two possible solutions: the trivial $v_{i\infty}(N)=0$, corresponding to the exact absorbing state in SIS epidemics, and the non-trivial solution, corresponding to the metastable SIS regime. In this paper, we focus on the metastable SIS regime.

The infection probabilities could be substantially different after some nodes decide to invest in a protection, causing those nodes not to be part of the epidemic process. Three scenarios are considered. The first one is a single community game, which could be regarded as a simple social network or a wireless and other full mesh networks (e.g., MANETs). We also study bipartite networks, often employed in the design of telecommunication networks. The main reason is that a bipartite topology provides satisfactory level of robustness after node or link failures. For instance, the topology of the Amsterdam Internet Exchange is designed as a bipartite network such that all the locations in Amsterdam are connected to two high throughput Ethernet switches. In addition, the topologies of sensor networks are also bipartite graphs.

A. Single community (full mesh) network

We first consider a single community (or full mesh) network, modeled as a complete graph K_N, where $a_{ij}=1$ for all i and j. If some nodes are removed from the original graph, the resulting graph K_n is also a complete graph, where $n\in\{0,1,\ldots,N\}$. By symmetry, we have that all $v_{i,\infty}(n)$ are equal. For brevity, in such a case, we sometimes use the notation $v_{.,\infty}(n)=v_{i,\infty}(n)$ From (3), we have [1], [5],

$$v_{.,\infty}(n)=v_{i,\infty}(n)=\begin{cases}1-\frac{1}{\tau(n-1)}, & \text{if}\quad \tau\geq\frac{1}{n-1},\\ 0, & \text{otherwise}\end{cases} \tag{4}$$

for each node i in a complete graph.

B. Bipartite network

The bipartite network $K_{M,N}$ consists of two clusters, $\mathcal{M}$ and $\mathcal{N}$, with M and N nodes, such that one of the nodes within a cluster are connected, but all the nodes from different clusters are connected to each other. Therefore, there are exactly $(M+N)$ nodes and MN links in the network.

The bipartite network also possesses an interesting property: if nodes from any of the two clusters are removed from the original graph, the resulting graph is again a bipartite graph $K_{m,n}$, where m and n are the number of remaining nodes in $\mathcal{M}$ and $\mathcal{N}$ clusters, respectively.

The governing equations (3) reduce for $K_{N,M}$ to two equations with two unknowns - the infection probabilities $v_{.,\infty}^{(\mathcal{M})}(m,n)$ and $v_{.,\infty}^{(\mathcal{N})}(m,n)$ in the clusters $\mathcal{M}$ and $\mathcal{N}$, respectively, in the metastable state stationary regime. Contrarily to the complete graph, the two probabilities depend on two values: m and n. The solution is [1],

$$v_{.,\infty}^{(\mathcal{M})}(m,n)=\frac{\tau^2mn-1}{\tau m(\tau n+1)}\text{ and }v_{.,\infty}^{(\mathcal{N})}(m,n)=\frac{\tau^2mn-1}{\tau n(\tau m+1)} \tag{5}$$

III. Game model on a single community network

In the investment game on the complete graph K_N, each node is a player and decides individually to invest in antivirus protection. The investment cost is C, while the infection cost is H. When a node invests, it is assumed to be directly immune to the virus and not part of the epidemic process anymore. Hence, this node cannot infect other nodes nor can be infected. If a node does not invest in antivirus protection, it is prone to the epidemics and might be infected by the virus (with rate β), but also can use additional protective mechanisms, like recovery or anti-spyware software (with rate δ). The induced network, without the nodes that decide to invest, is also a complete graph and it influences on the epidemic spread process.

A. Pure strategies

The investment cost for any player is a constant C and does not depend on the action of the other players. If a player decides not to invest, his cost is a linear function of its infection probability $v_{i,\infty}(n)$ of node i in the metastable state of the SIS process. The probability $v_{i,\infty}(n)$ depends explicitly on the number of nodes n that decide not to invest. In other words, there is an initial contact graph $G = K_N$ in which all the nodes are connected and the decisions of all the nodes induce an overlay graph $G_g = K_n$ only composed of the nodes that have decided *not* to invest.

1) Congestion Game: Due to players' decisions, we have a congestion game, because the utility of each player depends on the number of players that have decided not to invest. Each node has the choice between two actions: invest (further denoted by 1) or not (further denoted by 0). The payoff obtained by a player, in case he does not invest, depends on the number of players that choose the same action (0) not to invest. We denote by $\sigma_i \in \{0, 1\}$ the action of node i. For example, the payoff S_{i1} of a player $i \in \{1, 2, \ldots, N\}$ which decides to invest is defined by: $S_{i1} = C := S_1$, while the payoff of a player i which decides not to invest is: $S_{i0}(n) = Hv_{i,\infty}(n) := S_0(n)$. This game is a congestion game [27] as the payoff of a player depends on the number of players that choose his action. In the context of a congestion game, a (pure) Nash equilibrium is a vector of (pure) strategies, characterized by the number of nodes n^* that do not invest. We remark that several Nash equilibria lead to the same n^*. We are interested in the existence and uniqueness of this value n^*.

Definition 1. *At a Nash equilibrium, no node has an interest to change unilaterally his decision. The number n^* of nodes that do not invest at a Nash equilibrium is defined for any player i, by: $S_{i1} \leq S_{i0}(n^* + 1)$ and $S_{i0}(n^*) \leq S_{i1}$.*

Our game is symmetric as all players share the same set of payoff functions. The following important property (in Proposition 1) says that our game is not only a congestion game but also a potential game, due to the potential formula in [28, Theorem 3.1].

Proposition 1. *The game is* potential*, where $\Phi(n) = C(N - n) + H \sum_{k=2}^{n} v_{.,\infty}(k)$ is the potential function of the game.*

The existence of a potential function in a game shows the existence of pure Nash equilibrium: any minimum of the potential function Φ is a pure equilibrium. The existence also allows decentralized procedures like best response dynamics or reinforcement learning [29], [30] to converge to the pure Nash equilibrium. We can assume, for example, that an investment is valid only for a fixed amount of time and then each node pays again after expiration of his license.

Proposition 2. *For the number of nodes n^* that do not invest at equilibrium, the following inequality holds:*

$$v_\infty(n^*) \leq \frac{C}{H} \leq v_\infty(n^* + 1).$$

Moreover, above the epidemic threshold ($\tau > \frac{1}{N-1}$), n^* is uniquely defined by:

$$n^* = \begin{cases} \min\left\{N, \lceil \frac{1}{(1-\frac{C}{H})\tau} \rceil\right\}, & \text{if } C < H \\ N, & \text{otherwise} \end{cases} \tag{6}$$

where $\lceil x \rceil$ is the closest integer greater or equal than x and N is the total number of nodes.

2) Performance of the equilibrium: In order to evaluate the performance of the system, considering a non-cooperative behavior of each node, we use the Price of Anarchy (PoA) metric [20]. We define the social welfare $SW(n)$ of this system, when n users do not invest, as the summation of the cost for all users:

$$SW(n) = \sum_{i=1}^{N} S_{i\sigma_i}(n) = C(N - n) + nHv_{.,\infty}(n) \tag{7}$$

We define n^{opt} such that: $n^{opt} = \arg\min_n SW(n)$, while the Price of Anarchy, considering pure strategies, is defined by:

$$PoA_p = \frac{SW(n^*)}{SW(n^{opt})} \geq 1.$$

Before finding the Price of Anarchy, we first need to characterized the globally optimal solution solution.

Proposition 3. *The value that minimizes the social welfare is $n^{opt} \in \{N, \lceil 1 + \frac{1}{\tau} \rceil\}$.*

The following Corollary 1 also holds:

Corollary 1. *The equilibrium value n^* is at least as large as the optimum value n^{opt}, thus $n^* \geq n^{opt}$.*

We have determined n^* and n^{opt}, therefore, we can find PoA_p in an exact, but rather complex form. However, we can obtain a simple upper bound for PoA_p.

Corollary 2. *The Price of Anarchy PoA_p is bounded by:*

$$1 \leq PoA_p \leq \frac{1}{1 - (1 + \frac{1}{\tau})\frac{1}{N}}.$$

B. Symmetric mixed strategies

We now assume that each individual decides with a probability p to invest in the anti-virus protection. Moreover, the game is symmetric and then we look for a symmetric mixed Nash equilibrium. Each individual is faced with a new game, which depends on the realization of the random choice process of all the other individuals. We denote by $\bar{S}_i(\sigma_i, p)$ the expected cost of player i choosing the pure strategy σ_i against the probability choice p of the other $N-1$ players. For any user i, we have $\bar{S}_{i\sigma_i}(p) = \sum_{n=0}^{N-1} S_{i\sigma_i}(n + 1)\binom{N-1}{n}(1-p)^n p^{N-1-n}$, where by definition $S_{i0}(1) = 0$. Hence, the total expected cost of node i which invests with probability p' and when all the other nodes invest with probability p, is:

$$\bar{S}_i(p', p) = p'\bar{S}_{i1}(p) + (1 - p')\bar{S}_{i0}(p). \tag{8}$$

Definition 2. *At equilibrium, the indifference property p^* is solution of $\bar{S}_i(0, p^*) = \bar{S}_i(1, p^*)$.*

Definition 2 is a starting point for the characterization of the mixed equilibrium. The existence and uniqueness of a symmetric mixed equilibrium p^* are shown by Propositions 4 and 5, respectively.

Proposition 4. *A symmetric mixed equilibrium exists.*

Now, the equilibrium point p^* could be determined from an exact, but rather complex, non-closed expression in p:

$$\begin{aligned}\bar{S}_i(0,p) &= \sum_{n=0}^{N-1} S_{i0}(n+1)\binom{N-1}{n}(1-p)^n p^{N-1-n}\\ &= H\sum_{n=\underline{n}}^{N-1}(1-\frac{1}{\tau n})\binom{N-1}{n}(1-p)^n p^{N-1-n}, \quad (9)\end{aligned}$$

with $\underline{n} = \lceil \frac{1}{\tau} \rceil$ because $S_{i0}(n+1) = v_{i,\infty}(n+1)$ if $\tau \geq \frac{1}{n}$ and $S_{i0}(n+1) = 0$, otherwise.

Proposition 5. *The symmetric mixed equilibrium is unique.*

Expression (9) involves *generalized hyper-geometric functions* [31], which explains the difficulty of finding a closed form for p^*.

1) Approximation: In order to get a closed form expression of the symmetric mixed strategy, we consider the following approximation: instead of considering a player faced to realize a symmetric mixed strategy of the other players and optimizing his average cost, we consider that a player is part of an average game. If player i chooses strategy 1 with probability p' we obtained the following average approximated cost:

$$\hat{S}_i^{\text{approx}}(p',p) = p'C + (1-p')Hv_{i,\infty}(\bar{n}(p)+1), \quad (10)$$

where $\bar{n}(p)$ is the average number of nodes, except node i, that decide not to invest, i.e. $\bar{n}(p) = (1-p)(N-1)$. Based on Bernstein theorem [32], one can find that: for an arbitrary small $\varepsilon > 0$, there exists $n_0(\varepsilon, N, \tau, p, p')$, such that for each $N \geq n_0(\varepsilon, N, \tau, p, p')$, it holds $\mid \hat{S}_i^{\text{approx}}(p',p) - \bar{S}_i(p',p) \mid < \varepsilon$. In other words, $\hat{S}_i^{\text{approx}}(p',p)$ can be arbitrary close, for any ε, to the real $\bar{S}_i(p',p)$. Moreover, numerical simulations in Fig. 1a show that the corresponding PoA's are similar even for low N. Finally, Proposition 6 characterizes the mixed equilibrium, based on the approximation.

Proposition 6. *If we approximate the number of nodes that do not invest by its average, we obtain the following symmetric mixed Nash equilibrium:*

$$\hat{p}^* = \begin{cases} 1 - \frac{H}{\tau(H-C)(N-1)}, & \text{if} \quad C < H(1-\frac{1}{\tau(N-1)}),\\ 0, & \text{otherwise.}\end{cases}$$

If the investment cost C is higher than the curing cost H, then the equilibrium is $\hat{p}^* = 0$, because even, if a node is infected, its cost H is less than the cost C, then he would pay to be protected.

2) Performance of symmetric mixed equilibrium: The social welfare can be defined considering the mixed strategies: $SW(p) = \sum_{i=1}^{N} \bar{S}_i(p,p)$.

Further, we compute the optimal social welfare by using the approximated cost function $\hat{S}_i$:

$$\arg\min_p SW(p) \simeq \arg\min_p \left[p\hat{S}(1,p) + (1-p)\hat{S}(0,p)\right]$$

$$= \arg\min_p \begin{cases} (C-H)p + H(1-\frac{1}{\tau(N-1)}), & \text{if} \quad p \in [0, 1-\frac{1}{\tau(N-1)})\\ pC, & \text{if} \quad p \in [1-\frac{1}{\tau(N-1)}, 1].\end{cases}$$

Proposition 7. *The optimal solution of the social welfare (SW) is* $N \cdot \min\{C,H\}(1-\frac{1}{\tau(N-1)})$ *and it is achieved for*

$$\hat{p}^{opt} = \begin{cases} 0, & \text{if} \quad C > H\\ [0, 1-\frac{1}{\tau(N-1)}], & \text{if} \quad C = H\\ 1-\frac{1}{\tau(N-1)}, & \text{if} \quad C < H.\end{cases}$$

Based on Proposition 7 and $SW(p^*) = NC$, we are now able to approximate PoA_m in the case of mixed strategies:

Corollary 3. *When each node uses a mixed strategy, the Price of Anarchy* PoA_m *can be approximated by:*

$$PoA_m \approx \frac{C}{\min\{C,H\}(1-\frac{1}{\tau(N-1)})}.$$

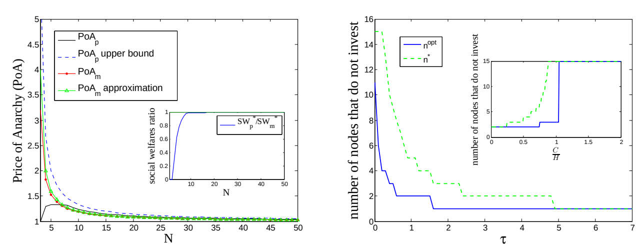
C. Comparison of strategies

In the previous sections, we have studied two different approaches for our non-cooperative investment game: the pure and the mixed strategies. These two game variants assume significantly different decision processes for each node. First, the approximation of the expected number of nodes that do not invest at equilibrium is very close to the result obtained using the potential game approach: $\hat{n} = N(1-\hat{p}^*) \simeq n^*$. Second, we compare the social welfares obtained in each situation, and we observe that pure strategies always yields a lower social welfare compared with symmetric mixed strategies.

As stated in Corollary 3, $\bar{S}_i(0,p^*) = \bar{S}_i(1,p^*) = C$, i.e. the payoff of all the players is equal, hence $SW_m^* = CN$. On the other hand, in the proof of Corollary 2, we find $SW_p^* := SW(n^*) > CN$. Corollary 4 immediately follows.

Corollary 4. *The social welfare is smaller if all the nodes use a pure strategy* (SW_p^*) *compared to the case in which all the nodes use a symmetric mixed strategy* (SW_m^*)*, i.e.* $SW_p^* < SW_m^*$.

The bound achieved in Corollary 4 is tight, because $(SW(n^*) - CN)$ is small - based on Proposition 2. This is also visualized in Fig. 1a, where indirectly, by comparing the Price of Anarchy for different equilibria, we show the approximation leads to almost correct value for the real expected payoff.



(a) (b)

Fig. 1: (a) Price of Anarchy depending on the number of nodes N (main plot). Ratio $\frac{SW_p^*}{SW_m^*}$ of the social welfares depending on the size N of the network (inset); (b) Number of nodes which do not invest as a function of τ (main plot) or the ratio $\frac{H}{C}$ (inset) for $N = 15$.

We evaluate the performance of the decentralized system (equilibrium) compared to the centralized point of view (social optimum) via the Price of Anarchy of our system in different cases: pure and mixed strategies. We show how this metric depends on the system parameters, such as the number of nodes (decision makers), the effective epidemic spreading rate $\tau = \frac{\beta}{\delta}$ and the costs C and H.

Fig. 1a illustrates the PoA with the following costs $C = 0.4$, $H = 0.5$ and the effective spreading rate $\tau = 2/3$. We observe that when the number of nodes is relatively small ($N < 8$): using pure strategies yields a smaller PoA compared to the case of mixed strategies. Moreover, we find that the upper bound of the pure PoA_p is very close to both PoA_p and PoA_m, when N becomes relatively large ($N > 10$). We also observe that the approximation of the expected payoff, which induces a closed form expression of the mixed equilibrium, is very close to the exact PoA_m. We show in Fig. 1a (inset) the ratio $\frac{SW_p^*}{SW_m^*}$ depends on the size N of the network. Fig. 1a matches Corollary 4, i.e., the social welfare obtained using pure strategies in the game, is lower than the one obtained via symmetric mixed strategies. This difference is noticeable when the network is small but diminishes quickly (e.g., for $N = 8$, $\frac{SW_p^*}{SW_m^*} = 0.9821$, and for $N \geq 10$, $\frac{SW_p^*}{SW_m^*} \approx 1$).

In Fig. 1b, we describe the number of nodes which do not invest considering the two methods: decentralized n^* (Nash equilibrium) and the centralized case n^{opt} (social welfare), depending on the effective spreading rate τ (main plot) and ratio of the costs of not investing and investing $\frac{H}{C}$ (inset). First, we observe that our result is correct, i.e., considering a decentralized point of view, the number of nodes which invest is lower than that of the centralized point of view. This result is somewhat surprising, as in general in a decentralized system, the players are more suspicious and we would think that in our setting, more nodes would invest at equilibrium compared to the central decision. Second, those numbers are exponentially decreasing with the effective spreading rate τ: the more the infection rate β dominates the curing rate δ, more nodes decide to invest in equilibrium. On the other hand, the number of nodes increases if the relative cost of investment decreases, as expected. However, the increase is faster in a decentralized system for a fixed $\frac{H}{C}$ (Fig. 1b inset).

IV. Game model in bipartite network

In this section, we characterize the equilibrium points, their existence and uniqueness for bipartite network $G_{M,N}$. If m and n nodes do not invest in an anti-virus, from partitions $\mathcal{M}$ and $\mathcal{N}$, respectively, the induced graph is also bipartite $G_g = K_{m,n}$. For simplicity, we define $k = \frac{C}{H}$.

Proposition 8. *The equilibrium pair (n^*, m^*) exists and satisfies the following inequalities. For each node*

$$\text{from } \mathcal{M},\ v_{\cdot,\infty}^{(\mathcal{M})}(n^*, m^*) < \frac{C}{H} \leq v_{\cdot,\infty}^{(\mathcal{M})}(n^*, m^*+1) \text{ and}$$

$$\text{from } \mathcal{N},\ v_{\cdot,\infty}^{(\mathcal{N})}(n^*, m^*) < \frac{C}{H} \leq v_{\cdot,\infty}^{(\mathcal{N})}(n^*+1, m^*)$$

Moreover, above the epidemic threshold, the following hold:

1) *for a given n^* (m^*) there is no more than one m^* (n^*).*
2) *for any τ and $k \geq \frac{1}{2}$; or $\tau \geq \frac{(1+k)(1-2k)}{2k(1-k)}$ and $k < \frac{1}{2}$: $|n^* - m^*| \leq 1$ i.e. n^* and m^* are either equal or consecutive integers.*
3) *in general, it is possible to have multiple equilibria pairs such that $|n^* - m^*| \geq 2$ for some (n^*, m^*).*

The social welfare is now given by:

$$SW(n,m) = \sum_{i=1}^{N} S_{i\sigma_i}(n) = C(N - n + M - m) + H(nv_\infty^n(n,m) + mv_\infty^m(n,m)). \quad (11)$$

We define the optimal pair $(n^{opt}, m^{opt}) = \arg\min_{(n,m)} SW(n,m)$ and the Price of Anarchy: $PoA := \frac{SW(n^*,m^*)}{SW(n^{opt},m^{opt})}$.

Proposition 9. *In $K_{N,M}$, the minimum (optimal) value of the social welfare is equal to $SW = \max\{\tau^2 MN - 1, 0\} \cdot \min\{\frac{C}{\tau^2 \max\{M,N\}}, H\frac{\tau(M+N)+2}{\tau(\tau M+1)(\tau N+1)}\}$. In particular,*

1) *if $MN \leq \frac{1}{\tau^2}$, then $SW = 0$ and $(n^{opt}, m^{opt}) = (N, M)$.*
2) *if $MN > \frac{1}{\tau^2}$, $\tau \max\{M,N\}\frac{\tau(M+N)+2}{(\tau M+1)(\tau N+1)} \geq k$ then: $SW = C\frac{\tau^2 MN-1}{\tau^2 \max\{M,N\}}$ and $(n^{opt}, m^{opt}) = (\frac{1}{\tau^2 M}, M)$ if $M > N$; $(n^{opt}, m^{opt}) = (N, \frac{1}{\tau^2 N})$ if $M < N$ or both points for $M = N$.*
3) *if $MN > \frac{1}{\tau^2}$, $\tau \max\{M,N\}\frac{\tau(M+N)+2}{(\tau M+1)(\tau N+1)} < k$ then $SW = H\frac{(\tau^2 MN-1)[\tau(M+N)+2]}{\tau(\tau M+1)(\tau N+1)}$ and $(n^{opt}, m^{opt}) = (N, M)$.*

Based on the results in Propositions 8 and 9, we can now find a tight bound for the Price of Anarchy (PoA).

Corollary 5. *The Price of Anarchy is bounded by:*

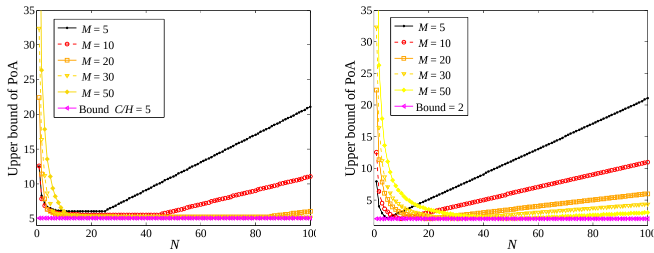
$$PoA \leq \frac{\tau(M+N)}{\max\{\tau^2 MN - 1, 0\}\min\{\frac{1}{\tau\max\{M,N\}}, \frac{H(\tau(M+N)+2)}{C(\tau M+1)(\tau N+1)}\}}.$$

The only used inequality in the proof of Corollary 5 is from Proposition 8. Moreover, we have the following Corollary 6.

Corollary 6. *The upper bound in Corollary 5 $\frac{\tau(M+N)}{\max\{\tau^2 MN-1,0\}\min\{\frac{1}{\tau\max\{M,N\}}, \frac{H(\tau(M+N)+2)}{C(\tau M+1)(\tau N+1)}\}}$ is bigger than $\max\{2, \frac{C}{H}\}$.*

Because the bound of PoA from Proposition 5 is accurate, Corollary 6 tells us that the loss of the social welfare due to decentralized investment decision often could be larger than 100% from the optimal. For a bipartite graph, not much could be said about the mixed equilibrium due the fact that the bipartite network is not symmetric, and players' uniform social welfare function cannot be defined.

For the bipartite network, the upper bound of the Price of Anarchy (PoA) is illustrated in Fig. 2. Figs. 2a and 2b both demonstrate the change of the upper bound of the Price of Anarchy as a function of N for several fixed values of M. All the figures confirm Corollary 6 that the upper bound of PoA is bigger than the maximum of 2 or $\frac{C}{H}$. In all the



(a) $\frac{C}{H} = 5$ (b) $\frac{C}{H} = 0.2$

Fig. 2: 2D plots of the upper bound of the Price of Anarchy as functions of N for fixed M.

cases, the closer M and N are to one another - the smaller upper bound of PoA (the minimum values in Figs. 2a and 2b). For fixed M and $\frac{C}{H} < 2$, the upper bound is dominated by $\frac{\tau^2(M+N)\max\{M,N\}}{\tau^2 MN-1}$ (Corollary 5), which decreases in N for $N < M$, achieves its minimum (close to 2) and then increases for $N > M$ (Fig. 2b). For fixed M and $\frac{C}{H} > 2$, the upper bound is dominated by $\frac{C}{H}\frac{\tau(M+N)(\tau M+1)(\tau N+1)}{(\tau^2 MN-1)(\tau(M+N)+2)}$ (Corollary 5), which decreases in N for $N < M$; achieves its minimum (close $\frac{C}{H}$); stays almost constant for $M \approx N$; and increases for $N > M$ (Fig. 2a).

V. Conclusions

In this paper, we explore the problem of finding optimal decentralized protection strategies in a network, where a node decides to invest in an anti-virus or to be prone to the virus SIS epidemic spread process. If a node (host) decides to invest, it cannot be infected, while if a node chooses not to invest, it can be infected by a virus and further spreads the virus inside the network. We study this problem from a game theoretic perspective. If a node decides to invest, the cost function of the node is the investment cost, otherwise the cost function is linearly proportional to the node's infection probability in the epidemic steady state.

We show the existence of a potential structure, which allows us to prove the existence and uniqueness and derive the pure and mixed equilibrium in a *single-community* (or *mesh*) network. Moreover, we find the pure equilibrium in a *bipartite* network. We also evaluate the performance of the equilibrium by finding the Price of Anarchy (PoA).

References

[1] P. Van Mieghem, J. Omić, and R. Kooij, "Virus spread in networks," *IEEE/ACM Trans. on Netw.*, vol. 17, no. 1, pp. 1 –14, feb. 2009.

[2] D. Chakrabarti, Y. Wang, C. Wang, J. Leskovec, and C. Faloutsos, "Epidemic thresholds in real networks," *ACM Trans. Inf. Syst. Secur.*, vol. 10, no. 4, pp. 1:1–1:26, Jan. 2008.

[3] A. Ganesh, L. Massoulie, and D. Towsley, "The effect of network topology on the spread of epidemics," in *IEEE INFOCOM*, vol. 2, 2005, pp. 1455–1466 vol. 2.

[4] J. Kephart and S. White, "Directed-graph epidemiological models of computer viruses," in *IEEE Symp. on Research in Security and Privacy*, 1991, pp. 343–359.

[5] P. Van Mieghem, "The N-intertwined SIS epidemic network model," *Computing*, vol. 93, no. 2-4, pp. 147–169, 2011.

[6] H. Wang, Q. Li, G. D'Agostino, S. Havlin, H. E. Stanley, and P. Van Mieghem, "Effect of the interconnected network structure on the epidemic threshold," *Phys. Rev. E*, vol. 88, p. 022801, Aug 2013.

[7] D. Li, P. Qin, H. Wang, C. Liu, and Y. Jiang, "Epidemics on interconnected lattices," *EPL (Europhysics Letters)*, vol. 105, no. 6, p. 68004, 2014.

[8] Y. Wan, S. Roy, and A. Saberi, "Designing spatially heterogeneous strategies for control of virus spread," *IET Systems Biology*, vol. 2, no. 4, pp. 184–201, July 2008.

[9] V. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, and G. Pappas, "Optimal vaccine allocation to control epidemic outbreaks in arbitrary networks," in *IEEE Conference on Decision and Control (CDC)*, Dec 2013, pp. 7486–7491.

[10] C. Enyioha, V. Preciado, and G. Pappas, "Bio-inspired strategy for control of viral spreading in networks," in *ACM Intl. Conf. on High Confidence Netw. Syst. (HiCoNS)*. ACM, 2013, pp. 33–40.

[11] E. Gourdin, J. Omić, and P. Van Mieghem, "Optimization of network protection against virus spread," in *Intl. Workshop on the Design of Reliable Commun. Netw. (DRCN)*, 2011, pp. 86–93.

[12] R. Cohen, S. Havlin, and D. ben Avraham, "Efficient immunization strategies for computer networks and populations," *Phys. Rev. Lett.*, vol. 91, p. 247901, Dec 2003.

[13] J. Omić, J. Martín-Hernández, and P. Van Mieghem, "Network protection against worms and cascading failures using modularity partitioning," in *Intl. Teletraffic Congress (ITC)*, Sept 2010, pp. 1–8.

[14] M. Bloem, T. Alpcan, and T. Basar, "Optimal and robust epidemic response for multiple networks," *Control Engineering Practice*, vol. 17, no. 5, pp. 525 – 533, 2009.

[15] E. Altman, T. Jimenez, T. Basar, and N. Shimkin, "Competitive routing in networks with polynomial cost," in *IEEE INFOCOM*, vol. 3, 2000, pp. 1586–1593 vol.3.

[16] Y. A. Korilis and A. A. Lazar, "On the existence of equilibria in noncooperative optimal flow control," *JACM*, vol. 42, no. 3, pp. 584–613, May 1995.

[17] A. Nahir, A. Orda, and D. Raz, "Workload factoring with the cloud: A game-theoretic perspective," in *IEEE INFOCOM*, 2012, pp. 2566–2570.

[18] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," *Computer Science Review*, vol. 3, no. 2, pp. 65 – 69, 2009.

[19] T. Roughgarden and E. Tardos, "How bad is selfish routing?" *JACM*, vol. 49, no. 2, pp. 236–259, Mar. 2002.

[20] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani, *Algorithmic Game Theory*. Cambridge University Press, 2007.

[21] J. Omić, A. Orda, and P. Van Mieghem, "Protecting against network infections: A game theoretic perspective," in *IEEE INFOCOM*, 2009, pp. 1485–1493.

[22] J. Aspnes, K. Chang, and A. Yampolskiy, "Inoculation strategies for victims of viruses and the sum-of-squares partition problem," *Jour. of Computer and System Sciences*, vol. 72, no. 6, pp. 1077 – 1093, 2006.

[23] D. Acemoglu, A. Malekian, and A. Ozdaglar, "Network security and contagion," MIT/LIDS intern report, Tech. Rep., 2013.

[24] L. Jiang, V. Anantharam, and J. Walrand, "Efficiency of selfish investments in network security," ser. NetEcon. ACM, 2008, pp. 31–36.

[25] M. Lelarge and J. Bolot, "Network externalities and the deployment of security features and protocols in the internet," in *SIGMETRICS*. ACM, 2008, pp. 37–48.

[26] S. Trajanovski, Y. Hayel, E. Altman, H. Wang, and P. Van Mieghem, "Decentralized Protection Strategies against SIS Epidemics in Networks," *arXiv* (submitted to *IEEE Trans. Control of Network Systems*), 2014, [Online]. Available: http://arxiv.org/abs/1409.1730.

[27] R. Rosenthal, "A class of games possessing pure-strategy nash equilibria," *Intl. Journal of Game Theory*, vol. 2, no. 1, pp. 65–67, 1973.

[28] D. Monderer and L. S. Shapley, "Potential games," *Games and Economic Behavior*, vol. 14, no. 1, pp. 124 – 143, 1996.

[29] Y. Chen and R. Gazzale, "When does learning in games generate convergence to nash equilibria? the role of supermodularity in an experimental setting," *American Economic Review*, vol. 94, no. 5, pp. 1505–1535, September 2004.

[30] R. Cominetti, E. Melo, and S. Sorin, "A payoff-based learning procedure and its application to traffic games," *Games and Economic Behavior*, vol. 70, no. 1, pp. 71–83, September 2010.

[31] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, ser. Encyclopedia of Mathematics and its Appl. Cambridge Univ. Press, 2004.

[32] S. N. Bernstein, "Démonstration du théorème de weierstrass fondée sur le calcul des probabilités," *Comm. Soc. Math. Kharkov*, vol. 13, no. 1, pp. 1–2, 1912, (in French).