Nash Equilibria in Shared Effort Games*

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Abstract

Shared effort games model people's contribution to projects and sharing the obtained profits. Those games generalize both public projects like writing for Wikipedia, where everybody shares the resulting benefits, and all-pay auctions such as contests and political campaigns, where only the winner obtains a profit. In θ-equal sharing (effort) games, a threshold for effort defines which contributors win and then receive their (equal) share. (For public projects $\theta = 0$ and for all-pay auctions $\theta = 1$.) Thresholds between 0 and 1 can model games such as paper co-authorship and shared homework assignments. We study existence and efficiency of such games, to know what will happen in a given situation and where an intervention may be needed to improve the social welfare. First, we fully characterize the conditions for the existence of a pure-strategy Nash equilibrium for two-player shared effort games with close budgets and project value functions that are linear on the received contribution and prove some efficiency results. Second, since the theory does not work for more players, fictitious play simulations are used to show when such an equilibrium exists and what its efficiency is. The results about existence and efficiency of these equilibria provide the likely strategy profiles and the socially preferred strategies to use in real life situations of contribution to public projects.

1 Introduction

People often invest in projects and share their revenues. Some examples are contributions to Wikipedia, or paper co-authorship [3]. In several real-life situations, like bonus points from shared homeworks, only the contributors who contribute at least a given threshold of the maximum contribution receive some profit. We want to know when such a situation will settle in an efficient equilibrium, and when some intervention may be required.

To model the situation we have described, we now define *shared effort games* (also appeared in [1]) that consist of a set of players and a set of projects. Each player has a budget to split somehow between a predefined subset of projects. The utility of each player is the sum of the utilities that she obtains from each project. A project's value is a function of the investments of the players in a given project. We consider a specific variant of a shared effort game, called a *θ-sharing mechanism*. In this mechanism, the project's value is equally divided between all the users who contribute at least θ of the maximum bid to the project.

Contributing to projects where only the maximum contributor to a project obtains the project's value, has been considered in all-pay auctions. The efficiency was bounded in [1], but assuming some very specific conditions. However, there is no analysis of the existence of Nash Equilibria (NE) and their efficiency in the general effort sharing interaction. Therefore, we analyze the existence and the efficiency of equilibria, to recommend socially optimum behavior to the contributors. We first theoretically characterize existence of NE in a particular case and find its efficiency. To consider some other case, we generalize fictitious play[1] to shared effort games (which are infinite), and use them to find NE. To simulate this, we provide a best response algorithm. We consider only pure NE throughout the paper.

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[1] The original fictitious play was proposed by Brown [2].

2 Theory of Nash Equilibrium

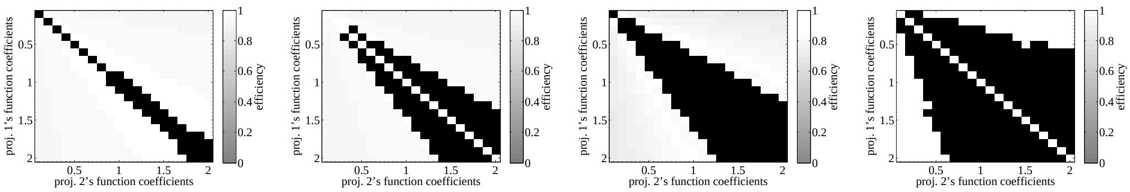
We study the existence of NE, and when it exists, we consider its efficiency. First, we prove some existence results for continuous sharing of projects' utilities. For the case of two players with close budgets and linear project functions, we characterize the existence of a NE. For a general n, some sufficiency conditions can be proven.

Since there may be various Nash Equilibria, we consider the price of anarchy (PoA), which is the ratio of the worst NE's efficiency to the optimum possible efficiency, and the price of stability (PoS), which is the ratio of the best NE's efficiency to the optimum possible efficiency. We prove that in the case that a NE exists in the above characterization, then almost always $\mathrm{PoA} = \mathrm{PoS} = 1$.

3 Simulations and Conclusions

To study when a game possesses at least one NE for some cases that are not yet handled, we generalize the fictitious play, originally suggested by Brown [2] for mixed extensions of finite games, and simulate it to find a NE. To simulate this, we devise an algorithm to find a best response to a given opponents' profile, if it exists. Any convergence provides a candidate for a NE, which is checked. Sometimes, our simulations find a NE for some games but we never assert that no NE exists.

Some results for the θ-equal 2-project case are presented in Fig. 1.



(a) $B_1 = 1, B_2 = 10, \theta = 0.2$
(b) $B_1 = 1, B_2 = 10, \theta = 0.5$
(c) $B_1 = 2, B_2 = 5, B_3 = 15, \theta = 0.5$
(d) $B_1 = 2, B_2 = 2, B_3 = 4, \theta = 0.2$

Figure 1: The simulation results for various thresholds θ and budgets B_i. Bold black color means that Nash Equilibrium has not been found. For all the other cases, the efficiency, a value in $[0, 1]$, is shown by the shade (from gray = 0.0 to white = 1.0).

For two players with far away budgets, a NE exists when the project function coefficients are not too close to one another. For $\theta = 0.5$, also equal coefficients bring upon a NE. For three players, a NE exists except, perhaps, when the project functions are quite close to each other.

The efficiency is close to optimum, except for the case of far away budgets, when the efficiency drops from somewhere in the interval $[0.6, 0.8]$ when the project function coefficients are quite close, to somewhere in the interval $[0.53, 0.68]$ when the project functions differ the most.

To summarize, the paper studied existence and efficiency of the NE in shared effort games, characterizing it for two players with close budgets, where the project functions are linear, and looking for NE through generalized fictitious play simulations for 2 projects. The found NE are optimum in almost all considered cases, besides in the case of three players with far away budgets.

Thus, this paper implies that for three or more players, some regulation may improve the total utility.

References

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